Midterm Solutions

1. Let G be a group and $H \lhd G$. State whether the following statements are true or false. Justify your answers.

   (a) If H abelian and G/H is abelian, then G is abelian.

   (b) If $N \lhd G$ and $N < H$, then $H/N \lhd G/N$.

   Solution. (a) This statement is **false.** A counterexample is D_{2n}, for $n \geq 3$, which is non-abelian. Note that $\langle r \rangle$ is an abelian subgroup of D_{2n} since it is cyclic, and $\langle r \rangle \lhd D_{2n}$ since $[D_{2n} : \langle r \rangle] = 2$. Moreover, $D_{2n}/\langle r \rangle \cong \mathbb{Z}_2$ since the only group of order 2 up to isomorphism is $\mathbb{Z}_2$), which is abelian.

   (b) This statement is **true.** First, note that since $N \lhd G$, it follows that $N \lhd H$, and so H/N is a group. Moreover, for arbitrary $gN \in G/N$ and $hN \in H/N$, we have:

   $$\begin{array}{rcll} (gN)(hN)(gN)^{-1} & = & (ghg^{-1})N & \text{(By definition of the operation in } G/N.) \\ & \in & H/N, & \text{(Since } H \lhd G, \text{ and so } ghg^{-1} \in H) \end{array}$$

   from which the assertion follows.

2. Let G, H be finite groups such that $G \ncong H$. Provide explicit non-trivial examples for the following.

   (a) A homomorphism $\varphi : G \to H$ such that $o(g) = o(\varphi(g))$, for all $g \in G$.

   (b) An epimorphism $G \to H$ when G is non-abelian and H is a non-cyclic abelian group.

   Solution. (a) By 3.2 (vii) of the Lesson Plan any monomorphism φ would serve as an example. In particular, we can consider the monomorphism $\varphi : C_n \to D_{2n}$ defined by $\varphi(e^{i2\pi k/n}) = r^k$, for $0 \leq k \leq n-1$ (Verify this!).

   (b) We know that $Z(D_8) = \{1, r^2\} \lhd D_8$ and $D_8/\{1, r^2\}$ is an abelian group of order 4 in which every nontrivial element is of order 2 (Verify this!). The quotient map $q : D_8 \to D_8/Z(D_8)$ is an epimorphism by 3.3 (i) of the Lesson Plan.

3. Given a group G, consider the set $\Delta = \{(x, x) : x \in G$.

   (a) Show that $\Delta < G \times G$.

(b) If G is abelian, use the First Isomorphism Theorem to show that

$$(G \times G)/\Delta \cong G$$

Solution. (a) Given arbitrary $(g,g) \in G \times G$ and $(h,h) \in G \times G$, we have:

$$\begin{array}{rcll} (g,g)(h,h)^{-1} & = & (g,g)(h^{-1},h^{-1}) & \text{(By definition of inverse in } G \times G.) \\ & = & (gh^{-1},gh^{-1}) & \text{(By definition of product in } G \times G.) \\ & \in & \Delta, & \text{(By definition of } \Delta) \end{array}$$

from which it follows that $\Delta < G \times G$ by the Subgroup Criterion (i.e. 1.2 (vii)) of the Lesson Plan.

(b) We consider the map $\varphi : G \times G \to G$ given by $\varphi(g,h) = gh^{-1}$, for all $(g,h) \in G \times H$.

φ is well-defined: Consider arbitrary $(g_1,h_1),(g_2,h_2) \in G \times G$ such that $(g_1,h_1) = (g_2,h_2)$. Then by definition of cartesian product, we have $g_1 = g_2$ and $h_1 = h_2$. Thus, it follows that

$$\varphi(g_1,h_1) = g_1h_1^{-1} = g_2h_2^{-1} = \varphi(g_2,h_2),$$

which shows that φ is well-defined.

φ is a homomorphism: For $(g_1,h_1),(g_2,h_2) \in G \times G$, we have:

$$\begin{array}{rcll} \varphi((g_1,h_1)(g_2,h_2)) & = & \varphi(g_1g_2,h_1h_2) & \text{(By definition of operation in } G \times G.) \\ & = & (g_1g_2)(h_1h_2)^{-1} & \text{(By definition of } \varphi.) \\ & = & g_1g_2h_2^{-1}h_1^{-1} & \text{(Basic properties of groups.)} \\ & = & (g_1h_1^{-1})(g_2h_2{-}1) & \text{(Since } G \text{ is ableian.)} \\ & = & \varphi(g_1,h_1)\varphi(g_2,h_1), & \text{(By definition of } \varphi.) \end{array}$$

which shows that φ is a homomorphism.

φ is surjective: Given any $g \in G$, we have $\varphi(g,1) = g(1)^{-1} = g$, which shows that φ is surjective.

$\ker\varphi$: We claim that $\ker\varphi = \Delta$. To see this, we have:

$$\begin{array}{rcll} \ker\varphi & = & \{(g,h) \in G \times H : \varphi(g,h) = 1\} & \text{(By definition of } \ker.) \\ & = & \{(g,h) \in G \times H : gh^{-1} = 1\} & \text{(By definition of } \varphi.) \\ & = & \{(g,h) \in G \times H : g = h\} & \text{(Basic properties of groups.)} \\ & = & \Delta, & \text{(By definition of } \Delta.) \end{array}$$

which establishes our claim. By the First Isomorphism Theorem, we have $G \times G/\ker\varphi \cong \operatorname{Im}\varphi$, from which the assertion follows.

4. Consider the additive group of rationals $\mathbb{Q}$.

   (a) Show that any cyclic subgroup of Q is of the from $x\mathbb{Z}$ for some $x \in \mathbb{Q}$.
   (b) Show that a finitely generated subgroup of $\mathbb{Q}$ is cyclic.
   (c) Given an example of a non-cyclic proper subgroup of $\mathbb{Q}$.

Solution. (a) Let $H = \langle \frac{p}{q} \rangle$ be a cyclic subgroup of the rationals, where we assume without loss of generality that $\frac{p}{q}$ is in the reduced form (i.e $\gcd(|p|, |q|) = 1$). Then :

$$\begin{array}{rcll} H & = & \{(\frac{p}{q})^k : k \in \mathbb{Z}\} & \text{(By definition of a cyclic group.)} \\ & = & \{\pm \sum_{i=1}^{k} \frac{p}{q} : k \in \mathbb{N} \cup \{0\}\} & \text{(By the operation in } \mathbb{Q}.) \\ & = & \{\pm k\frac{p}{q} : k \in \mathbb{N} \cup \{0\}\} & \text{(Basic additive arithmetic.)} \\ & = & \{\pm k\frac{p}{q} : k \in \mathbb{Z}\} & \text{(By definition of } \mathbb{Z}.) \\ & = & x\mathbb{Z}, \text{ where } x = \frac{p}{q}, & \text{(By definition of } x\mathbb{Z}.) \end{array}$$

and the assertion follows.

(b) Consider a finitely generated subgroup $H = \langle \{\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\} \rangle$. Since $\mathbb{Q}$ is abelian, any $h \in H$ is of the form

$$h = \sum_{i=1}^{r} k_i \frac{p_i}{q_i}, \text{ where } k_i \in \mathbb{Z}.$$

By simple arithmetic, it can be seen that the expression for h simplifies above to a fraction of the form $\frac{p}{q_1 q_2 \ldots q_r}$, for some $p \in \mathbb{Z}$ (Verify this!). Thus, it follows that

$$h = \sum_{i=1}^{p} \frac{1}{q_1 q_2 \ldots q_r} = (\frac{1}{q_1 q_2 \ldots q_r})^p.$$

In other words, $h \in \langle \frac{1}{q_1 q_2 \ldots q_r} \rangle$, which shows that H is cyclic.

(c) For a fixed prime number p, consider the subset

$$A_p = \{\frac{q}{p^k} : q, k \in \mathbb{Z}\}.$$

For arbitrary $\frac{q_1}{p^{k_1}}, \frac{q_2}{p^{k_1}} \in \mathbb{Q}$, we have:

$$\frac{q_1}{p^{k_1}} - \frac{q_2}{p^{k_2}} = \frac{q_1 p^{k_2} - q_2 p^{k_1}}{p^{k_1 + k_2}} \in A_p,$$

which shows that $A_p < \mathbb{Q}$ by the Subgroup Criterion.

5. **(Bonus)** Show that there can exist at most two non-abelian groups of order 8 up to isomorphism.

Solution. Let G be a group of order 8. By the Lagrange's Theorem any non-trivial element in G is of order 2, 4, or 8. If G has an element of order 8, then $G \cong \mathbb{Z}_8$, which is abelian. Thus, we have our first inference:

Inference 1. If G is non-abelian it cannot have an element of order 8.

Suppose that every non-trivial element of G is order 2. Then $h^2 = 1$ for every non-trivial $h \in G$, and so by Problem 1 of Quiz 1, it follows that G is abelian. So, G has to have a non-trivial element x with $o(x) = 4$. It follows immediately from 1.2 (vii) that $o(x^3) = 4$. Thus, we have second inference:

Inference 2. If G is non-abelian, then G has to have at least two elements of order 4 (namely $\{x, x^3\}$).

Consider a $y \in G \backslash N$. First, we show that y is distinct from the elements yx, yx^2, and yx^3. If $yx = y$, then $x = 1$, which is not possible since x is nontrivial. Also, $y = yx^2$, then $x^2 = 1$, which again contradicts the fact that $o(x) = 4$. Moroever, if $y = yx^3$, then $x^3 = 1$, which is again impossible since $o(x^3) = 4$. By similar arguments, we can show that that the yx^i, $i \in \{1, 2, 3\}$ are also distinct from each other. Thus, yx^i, $i \in \{1, 2, 3\}$, are distinct elements that are all distinct from y. This brings our to our third inference:

Inference 3. $G = \{1, x, x^2, x^3, y, yx, yx^2, yx^3\}$, which shows that $G = \langle x, y \rangle$.

We now consider the element yxy^{-1}. Since $\langle x \rangle \lhd G$, it follows that $yxy^{-1} \in \langle x \rangle$. Thus, yxy^{-1} equals one of 1, x, x^2, or x^3. Clearly $yxy^{-1} \neq 1$, for this would imply that $x = 1$, which is impossible. If $yxy^{-1} = x$, then $xy = yx$, and so the map $\psi : G \to \mathbb{Z}_4 \times \mathbb{Z}_2$ defined by $\psi(y^i x^j) = (j, i)$ is an isomorphism (Verify this!). This would imply that G is abelian. Moreover, $yxy^{-1} \neq x^2$ since $o(yxy^{-1}) = o(y) = 4$ (Verify this!), but $o(x^2) = 2$. From the preceding discussion, we have our fourth inference.

Inference 4. If G is non-abelian, then $yxy^{-1} = x^3 = x^{-1}$.

Finally, we consider $o(y)$; it is apparent that $o(y) = 2$ or 4. First, we consider the case when $o(y) = 2$. Then, we have:

$$\begin{aligned} (yx)^2 &= yxyx \\ &= (yxy^{-1})x \quad \text{(Since } o(y) = 2,\ y = y^{-1}.\text{)} \\ &= x^{-1}x = 1, \quad \text{(By Inference 4.)} \end{aligned}$$

which shows that $o(yx) = 2$. By similar arguments, we can show that $o(yx^2) = o(yx^3) = 2$. Thus, the map $\psi : G \to D_8 = \langle r, s\rangle$ defined by $\psi(y^i x^j) = s^i r^j$ for $0 \le i \le 1$ and $0 \le j \le 3$ is an isomorphism. (Verify this!) This leads to the following inference:

Inference 5. A possible non-abelian group of order 8 (up to isomorphism) is D_8, and this possibility occurs when $o(y) = 2$.

Finally, we consider the case when $o(y) = 4$. Then:

$$\begin{aligned}
(yx)^4 &= yxyxyxyx \\
&= (yxy^{-1})(y^2x)(yxy^{-1})(y^2x) && \text{(Basic group properties.)} \\
&= (x^{-1})(y^2x)x^{-1}(y^2x). && \text{(By Inference 4.)} \\
&= x^{-1}y^4x. && \text{(Basic group properties.)} \\
&= 1, && \text{(Since } o(y) = 4.\text{)}
\end{aligned}$$

which shows that $o(yx) = 4$. By similar arguments, it can be shown that $o(yx^2) = o(yx^3) = 4$. Note that in this case $G \not\cong D_8$ since G has five elements of order 4, namely $\{x, x^3, y, yx^2, yx^3\}$, while D_8 has only 2.

Inference 5. The only other possibility for a non-abelian group of order 8 up to isomorphism (besides D_8) occurs when $o(y) = 4$. (Note that in this case $G \cong Q_8$, the *group of quaternions.* We will study this group further in the second half of the semester.)

Conclusion: The upshot of the arguments above is that there can be at most two non-ableian groups of order 8 up to isomorphism.